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Calculation of temperature field in a thin moving sheet heated with laser beam

R. Brockmann [a,*], K. Dickmann [a], P. Geshev [b,1], K.-J. Matthes [c,2]

[a] Department of Physical Engineering, Lasercenter, University of Applied Sciences Muenster, Stegerwaldstrasse 39, D-48565 Steinfurt, Germany

[b] Institute of Thermophysics of Russian Academy of Sciences, Lavrentyev Avenue 1, 630090 Novosibirsk, Russia

[c] Institute for Product Engineering/Welding Technology, Technical University of Chemnitz, D-09107 Chemnitz, Germany

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Abstract

A temperature field in a thin moving sheet heated with a laser beam is calculated. The power density in the beam is distributed according the Gauss function. Cooling effects caused by the free or forced convection in ambient gas is taken into account. The problem becomes two dimensional by averaging the temperature field over the sheet thickness. Two-dimensional integral Fourier transformation on space coordinates is applied to solve the problem. It gives an analytical solution of the problem in the Fourier space. The inverse Fourier transformation is fulfilled and the solution is represented via an integral having exponential asymptotes at infinity.

Keywords: Heat Transfer; Laser; Thermophysical

1. Introduction

For effective laser material processing it is often useful to get a theoretical estimation of expected process results. Especially the heating of metal sheets with a laser beam requires to know the expected temperature field arising in the work piece depending on parameters like laser power, sheet velocity, gas flow and so on. This problem occurs for material processing like hardening, cutting or welding. The aim of this paper is to develop a simple algorithm for calculation of the laser induced temperature field in a thin moving sheet and to find an asymptotic formula for temperature at large distance from heating point. The asymptotic solution is compared with the full simple calculation, that does not consider non-linear effects like melting or evaporation, and it will be useful as source for a more complex model of this task, where also the Stefan problem will be considered.

A lot of finite element based solutions for special applications can be found in literature [1–6], but for this first simple linear model an analytical solution of the heat conduction equation is suggested. An analytical solution for the 3D temperature field arising around a point-like heat source moving through the medium was found by Rosenthal [7,8]. In the monograph [9] written later by Carslaw and Jaeger one can find more complex solutions for analogous problem in 2D space and also for a thin sheet with cooling effects due to ambient gas. These solutions were useful for estimating of the asymptote of the temperature fields for large distances from the heat source. Most former solutions of the heat conduction equation concerning laser beam welding are based on a semi-infinite geometry [10–13] and cannot be used in this special case of micro-welding of thin metal foils.

In a vicinity of the heat source the temperature field has a strong dependence on a shape of a heat distribution function. The temperature maximum can be reliably

* Corresponding author. Fax: +49-2551-962490.

E-mail addresses: laserlab@fh-muenster.de (R. Brockmann), geshev@ipt.nsc.ru (P. Geshev).

[1] Fax: +7-3832-357880.

[2] Fax: +49-371-5312441.

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calculated only if the heat distribution function is taken into account and correctly described. For the heating with a TEM_{00}-laser beam the natural shape of the heat distribution is the Gauss function. The Gauss distribution will be used below to describe the very local heating of thin moving sheets. Such kind of problem arises in micro-welding of metal thin sheets described in [14].

Ambient gas can take away a part of heat supplied to the sheet. Three mechanisms can give the cooling effect:

(a) the natural (free) convection,
(b) the forced convection (due to a blowing jet),
(c) the back heat irradiation from the hot sheet surface.

The first effect is usually small, but the next two ones must be included into the mathematical model. Below these cooling effects are considered in a traditional manner used in the heat transfer theory [2]: a heat transfer coefficient is introduced into the model. The developed model is validated for the case of beam radius greater than sheet thickness ($r_b > h$). Further on in this linear model the obedience of thermo-physical parameters on temperature and the heat of fusion or evaporation (Stefan problem) are not considered.

2. Setting of the problem and method of solution

Let us consider a thin sheet moving with a velocity u along the x'-axis. A laser beam propagating along the z'-axis illuminates the sheet from below (Fig. 1). A heat flux introduced into the sheet may be described by the Gauss function

$$q(r) = q_0 \exp\left(-r^2/r_b^2\right), \tag{1}$$

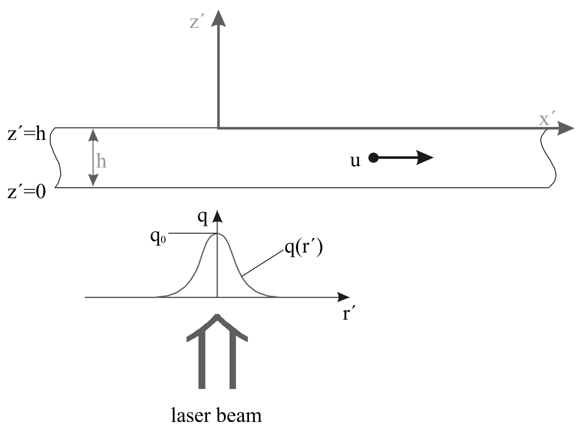

Fig. 1. A thin sheet is moving with a velocity u along the x'-axis. A laser beam propagating along the z'-axis illuminates the sheet from below.

where q_0 is the maximal heat flux density, r_b is the effective beam radius. These values determine the laser beam power I_0:

$$I_0 = \pi r_b^2 q_0. \tag{2}$$

In the moving sheet the temperature obeys the heat conductivity equation

$$\rho C_p u \frac{\partial T}{\partial x'} = \lambda\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)T, \tag{3}$$

where ρ, C_p, λ are the density, specific heat capacity, and heat conductivity, respectively. Boundary conditions have to account heat introduced in the sheet by the laser beam, $q(r)$ and also the heat leaving the sheet surfaces due to the cooling effect of ambient gas and back irradiation:

$$\text{at } z' = 0: \quad -\lambda\frac{\partial T}{\partial z'} = q(r) - \alpha_1(T - T_g), \tag{4}$$

$$\text{at } z' = h: \quad -\lambda\frac{\partial T}{\partial z'} = \alpha_2(T - T_g),$$

where α_1, α_2 are the heat transfer coefficients for the lower and upper surfaces of the sheet, T_g is the ambient gas temperature. The boundary condition at infinity for

$$r = \sqrt{x'^2 + y'^2} \to \infty \quad \text{has to be } T - T_g = 0. \tag{5}$$

It should be noted that the coefficients α have an empirical character and depend on the structure of gas flow at the sheet surface. For the case of natural (or free) convection α_2 is in the range of 10–100 W/m^2 K. For very intense flows (like in a blowing gas jet) one can expect values of $\alpha_1 \approx$ 100–5000 W/m^2 K.

If the sheet thickness is small and sheet is made of metal with a large heat conductivity λ one can neglect the temperature variations in z'-direction. By integration of Eq. (3) in z'-direction from $z' = 0$ to $z' = h$ and substituting the boundary conditions (4) in the Eq. (3) one can obtain

$$\rho C_p u h \frac{\partial \bar{T}}{\partial x'} = h\lambda\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right)\bar{T} + q(r) - (\alpha_1 + \alpha_2)(\bar{T} - T_g), \tag{6}$$

where $\bar{T}$ is the averaged temperature ($\bar{T} = \frac{1}{h}\int_0^h T\,dz'$). For Eq. (6) the same boundary condition (5) is valid.

By introducing the dimensionless variables

$$x = \frac{x'}{r_b}, \quad y = \frac{y'}{r_b}, \quad z = \frac{z'}{r_b}, \quad \theta = \frac{(\bar{T} - T_g)}{T_*},$$

$$T_* = \frac{I_0}{4\pi\lambda h} = \frac{q_0 r_b^2}{4\lambda h},$$

and the coefficients

$$Pe = \frac{\rho C_p u r_b}{4\lambda}, \quad Bi = \frac{\alpha r_b^2}{4\lambda h}.$$

(T_* is the characteristic temperature scale for the problem, Pe is the Peclet number, Bi is the Biot criterion based on the total heat transfer coefficient $\alpha = \alpha_1 + \alpha_2$) the governing Eq. (1) has the form

$$Bi\theta + Pe\frac{\partial\theta}{\partial x} - \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta = e^{-x^2-y^2}. \tag{7}$$

The next substitution

$$\theta(x,y) = e^{2Pex}\psi(x,y) \tag{8}$$

leads to another equation

$$(Bi + Pe^2)\psi - \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = e^{Pe^2-y^2-(x+Pe)^2}. \tag{9}$$

As it follows from Eqs. (8) and (9) the boundary conditions at infinity for function are zero. Indeed, if we suggest the substitution of

$$\psi = ce^{-\beta x} \quad (\beta \geqslant 0, \ \ c = \text{const.})$$

in Eq. (9), it follows:

$$\beta = 2\sqrt{Pe^2 + Bi}.$$

But from Eq. (8) follows, that $\beta < 2Pe$. This contradiction means, that the constant c is zero ($c = 0$) and ψ has zero limits in infinity for all directions.

As ψ in Eq. (9) vanishes in infinity, the two-dimensional integral Fourier transformation may be applied. After the transformation of this equation an analytic expression for the solution can be derived

$$\tilde{\psi}(\vec{k}) = \frac{\exp\left[Pe^2 + ik_xPe - k^2/4\right]}{4\pi(Bi + Pe^2 + k^2/4)}, \tag{10}$$

where $k^2 = k_x^2 + k_y^2$.

From (10) the original function $\psi(x,y)$ can be obtained as a twofold inverse Fourier transformation:

$$\psi(x,y) = \frac{1}{4\pi}\int\int_{-\infty}^{\infty} \frac{\exp\left[-k^2/4 + Pe^2 + ik_x(Pe + x) + ik_y y\right]}{Bi + Pe^2 + k^2/4}\,dk_x\,dk_y. \tag{11}$$

Let us introduce the vector $\vec{b} = (x + Pe, y)$ with $B = \|\vec{b}\| = (x + Pe)^2 + y^2$.

The integral (11) contains the scalar product $(\vec{k}\cdot\vec{b}) = kb\cos\varphi$, where φ is the azimuth angle in the Fourier space. Integrating about this angle φ from 0 to 2π according to Ref. [15] we get from formula (11)

$$\psi(x,y) = e^{Pe^2}\frac{1}{2}\int_0^{\infty}\frac{e^{-k^2/4}}{A + k^2/4}J_0(k\sqrt{B})k\,dk, \tag{12}$$

where $J_0(k\xi)$ is the Bessel function and $A = Bi + Pe^2$.

In the monograph [15] one can find the integral

$$I(s) = \frac{1}{2}\int_0^{\infty} e^{-sk^2/4}J_0(k\sqrt{B})k\,dk = \frac{e^{-B/s}}{s}. \tag{13}$$

In order to obtain from $I(s)$ the integral staying in the formula (12) the expression (13) must be multiplied by e^{-As}, and integrated on s from 1 to ∞.

$$\begin{aligned}\frac{1}{2}\int_0^{\infty}&\left(\int_1^{\infty} e^{-s(A+k^2/4)}\,ds\right)J_0(k\sqrt{B})k\,dk\\ &= \frac{1}{2}\int_0^{\infty}\frac{e^{-(A+k^2/4)}}{A + k^2/4}J_0(k\sqrt{B})k\,dk\\ &= \int_1^{\infty} e^{-(As+B/s)}\frac{ds}{s}.\end{aligned} \tag{14}$$

It follows from (14)

$$\begin{aligned}\frac{1}{2}\int_0^{\infty}&\frac{e^{-k^2/4}}{A + k^2/4}J_0(k\sqrt{B})k\,dk\\ &= e^A\int_1^{\infty} e^{-(As+B/s)}\frac{ds}{s}.\end{aligned} \tag{15}$$

The substitution of the integral (15) in the expression (12) leads to

$$\psi(x,y) = e^{Bi+2Pe^2}\int_1^{\infty} e^{-(As+B/s)}\frac{ds}{s}. \tag{16}$$

Finally one can obtain for the dimensionless temperature θ

$$\theta(x,y) = e^{Bi+2Pe(x+Pe)}\int_1^{\infty} e^{-(As+B/s)}\frac{ds}{s}, \tag{17}$$

where $A = Bi + Pe^2$ and $B = y^2 + (x + Pe)^2$.

The integral staying in (17) is a function of two variables A and B. This integral can be calculated via the series decomposition as shown in the Appendix A. An asymptotic formula for calculation θ at large values A and B is also described in Appendix A.

3. Results and discussion

The asymptotic formula for temperature at large x and y can be obtained with the help of formulae (A.15) and (A.16):

$$\theta_{as} = \frac{\sqrt{2\pi}}{F^{1/4}}\exp(Bi + 2Pe(x + Pe) - \sqrt{F}), \tag{18}$$

where $F = 4AB = 4(Bi + Pe^2)[y^2 + (x + Pe)^2]$.

At $y = 0$ and $(x + Pe) > 0$ the formula (18) gives exponential decay due to non-zero Biot criterion:

$$\begin{aligned}\theta_{as} = \frac{\sqrt{\pi}}{(Bi + Pe^2)^{1/4}(x + Pe)^{1/2}}&\exp(Bi - 2[(Bi + Pe^2)^{1/2}\\ &- Pe](x + Pe)).\end{aligned} \tag{19}$$

Only for $Bi = 0$ the last expression gives the known results, [2], the "minus 1/2 power" decay:

$$\theta_{as} = \sqrt{\frac{\pi}{Pe(x + Pe)}}, \quad (x + Pe) > 0. \tag{20}$$

The expressions (18)–(20) have a singularity at the point $y = 0$, $x = -Pe$. A small additive to the value F can avoid this singularity. This additive depends on A, so the improved expression for F is

$$F = (1 + 2A)\sqrt{A/2} + 4AB. \tag{21}$$

Note that in problems of laser micro-welding parameters Pe and Bi are small. They are significantly smaller than 1.

In Fig. 2(a) the profiles of functions θ and θ_{as} are shown for the parameters: $y = 0$; $Bi = 0.018$; $Pe = 0.1$. In Fig. 2(b) the parameter Pe has changed to $Pe = 1$.

One can see a long tail in these temperature profiles. The larger is the velocity of the sheet, the longer is the tail.

The position of the temperature maximum is described by the next equation

$$x_m + Pe = Pe\frac{\int_1^\infty \exp -[As + (x_m + Pe)^2/s]\frac{ds}{s}}{\int_1^\infty \exp -[As + (x_m + Pe)^2/s]\frac{ds}{s^2}}, \tag{22}$$

obtained from the expression (17). The maximal values of the function θ (depending on Bi and Pe) calculated at the point x_m is shown in Fig. 3(a) as a contour map.

Fig. 3(b) shows a demonstration of the map for dimensional temperature maximum at x_m: here the beam power is $I_0 = 10$ W, the sheet thickness is $h = 0.05$ mm, the specific heat is $C_p = 460$ J/kg K and heat conductivity of stainless steel is $\lambda = 14$ W/(K m).

From these maps the domain in parameter space can be established where melting of the sheet material occurs. For example the isotherm 1600 °C generates two inequalities where $T > 1600$ °C and melting appears: $Pe < 0.8$ and $Bi < 0.2$.

According to definitions (7) it means that in this case the beam radius and the sheet velocity obey the relations

$$r_b = \sqrt{0.8\frac{\lambda h}{\alpha}} = 334\ \mu\text{m} \quad \text{and} \quad u < 3.2\frac{\lambda}{\rho C_p r_b} \approx 126\ \text{mm/s} \tag{23}$$

(here α was taken equal to 5000 W/m^2 K).

According to formulae (23) one can find the parameters for optimizing the sheet heating, cutting or welding.

A temperature field arising around a point-like gauss-distributed heat source is exemplified in Fig. 4(a). The shown temperature field depends on dimensionless co-

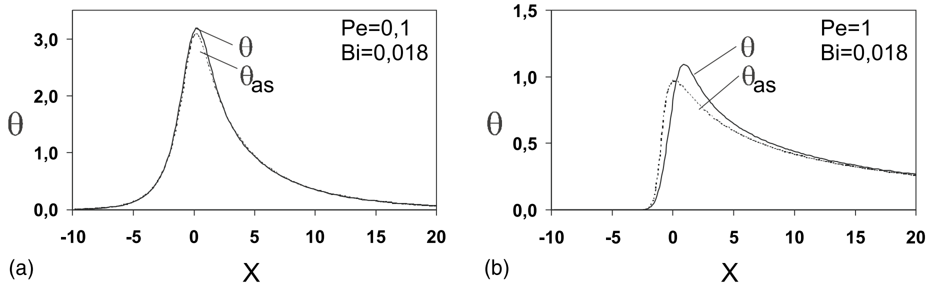

Fig. 2. Comparison of calculated and approximated functions: the solid curves are built according to expression (17) and (18).

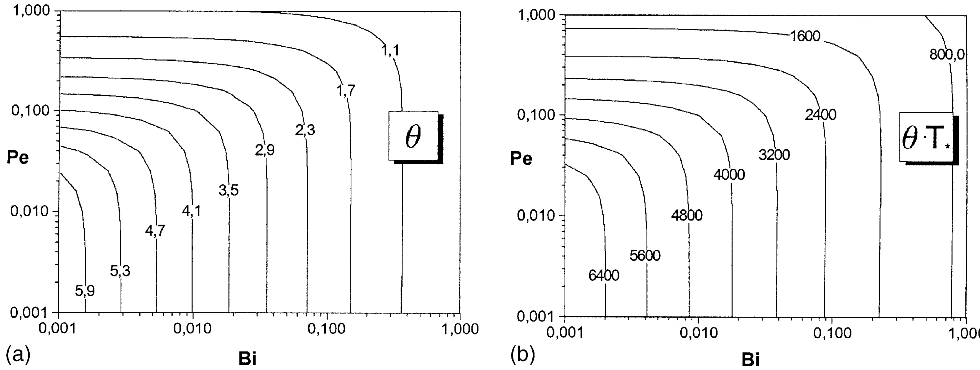

Fig. 3. (a) The contour map of function θ (at x_m) depending on Bi and Pe, (b) maximal dimensional temperature in °C (at x_m) depending on Bi and Pe.

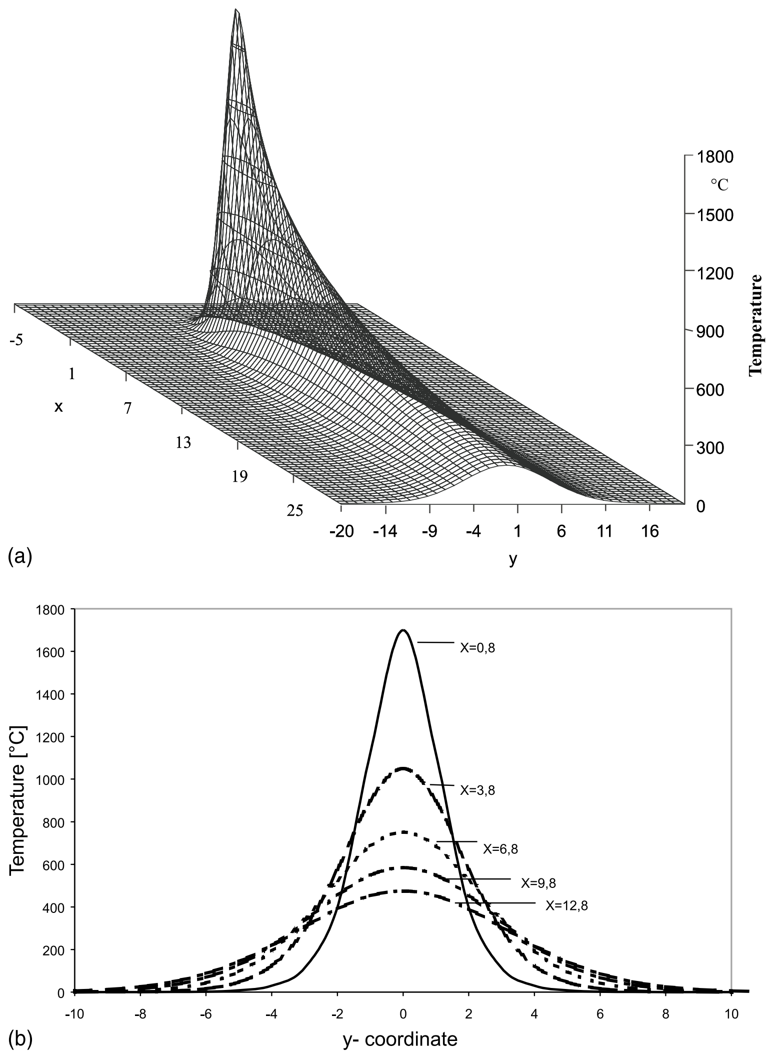

Fig. 4. (a) Dimensional temperature field over the dimensionless coordinates x and y, (b) selected temperature profiles along y-axis ($x = \text{const.}$) of temperature field shown in Fig. 3(a).

ordinates x and y is calculated for the following parameters:

Laser power P_l*:* 10 W,
Sheet thickness: 50 μm,
Spec. heat C_p*:* 460 J/kg K,
Heat transfer coefficient α*:* 5000 W/m^2 K,
Beam radius r_b*:* 100 μm,
Velocity u: 100 mm/s,
Heat conductivity λ*:* 14 W/K m.

To give a better expression of the temperature profile along the y-axis ($x = \text{const.}$), in Fig. 4(b) the profiles for selected x values ($x = 0.8$, 3.8, 6.8, 9.8 and 12.8) are extracted.

The maximum temperature is reached in point $x_m = 0.8$ and a high temperature gradient in y-direction is seen. A melting of material ($T > 1600$ °C) can be expected only in the middle zone a with width of $-r_b < y' < +r_b$ $(-1 < y < +1)$.

4. Conclusions

A mathematical model describing the temperature field on a thin moving sheet heated by a laser beam is developed. According to a TEM_{00} beam mode, the density of the heat flux supplied to the sheet surface is described by the Gauss formula. Cooling effects caused by gas flow and back irradiation are taken into account through the heat transfer coefficient. Small thickness of the sheet allows the averaging of the temperature field and the governing equation across the sheet. By applying the two-dimensional integral Fourier transformation the formula for the temperature field is built. An asymptote of the temperature at large values of coordinates is expressed in analytical form. A map of the maximal temperatures built as a function of Biot and Peclet criteria permits one to give an estimation, when the sheet material will be melted or even evaporated.

The asymptotic solution found in this article will be useful as the boundary conditions of a more complex model of the problem. In that forthcoming model, also non-linear effects will be taken into account (Stefan problem) and the geometry of the melting zone of laser micro-welding for thin metal foils will be predicted.

Appendix A

Let us consider the integrals

$$F(A,B) = \int_1^\infty e^{-(As+B/s)}\frac{ds}{s}, \tag{A.1}$$

$$f(A,B) = \int_0^1 e^{-(As+B/s)}\frac{ds}{s}. \tag{A.2}$$

A sum of these two integrals gives a full integral that can be represented in a form

$$F(A,B)+f(A,B) = \int_0^\infty e^{-(As+B/s)}\frac{ds}{s} = 2K_0(2\sqrt{AB}), \tag{A.3}$$

where $K_0(\xi)$ is the modified Bessel function (see [2]).

From the other side, the integral (A.2) can be transformed by the help of substitution $s = 1/t$. It leads to the expression

$$f(A,B) = -\int_\infty^1 e^{-(A/t+Bt)}\frac{ds}{s} = \int_1^\infty e^{-(Bt+A/t)}\frac{dt}{t}. \tag{A.4}$$

The comparison of expression (A.1) and (A.4) shows that $f(A,B)$ is the same integral as $F(A,B)$ but with permuted arguments:

$$f(A,B) = F(B,A). \tag{A.5}$$

Hence, from (A.3) follows,

$$F(A,B) = 2K_0(2\sqrt{AB}) - F(B,A). \tag{A.6}$$

This formula will be used below.

The exponential function $e^{-B/s}$ in the integral (A.1), can be represented as a series decomposition. After that the formula (A.1) takes the form

$$F(A,B) = \sum_{n=0}^\infty \frac{(-B)^n}{n!}I_{n+1}(A) \quad \text{for } B < A$$

and

$$F(A,B) = 2K_0(2\sqrt{AB}) - \sum_{n=0}^\infty \frac{(-A)^n}{n!}I_{n+1}(B) \quad \text{for } A < B, \tag{A.7}$$

where

$$I_{n+1}(A) = \int_1^\infty \frac{e^{-As}}{s^{n+1}}ds. \tag{A.8}$$

In the expression (A.7) the formula (A.6) is used.

The exponential integral I_{n+1} obeys the next recurrent formula

$$I_{n+1}(A) = \frac{e^{-A}}{n} - \frac{A}{n}I_n(A), \tag{A.9}$$

that permits all these integrals calculate via the first one $I_1(A)$. The integral

$$I_1(A) = \int_1^\infty \frac{e^{-As}}{s}ds \tag{A.10}$$

is calculated according to the algorithm described in the book [4].

In order to calculate $F(A,B)$ at large A and B we consider another representation of this integral

$$\begin{aligned} F(A,B) &= \int_{\xi_0}^\infty e^{-2\sqrt{AB}ch\xi}d\xi \\ &= K_0(2\sqrt{AB}) - \mathrm{Sign}(A-B)e^{-2\sqrt{AB}} \\ &\quad\times \int_0^{\tau_0}\frac{e^{-4\sqrt{AB}\tau}d\tau}{\sqrt{\tau(1+\tau)}}, \end{aligned} \tag{A.11}$$

where

$$\xi_0 = \ln\sqrt{A/B}, \quad \tau_0 = (Sh(\xi_0/2))^2 = \frac{A+B-2\sqrt{AB}}{4\sqrt{AB}}.$$

The expression (A.11) is applicable at $\tau_0 < 1$. A series decomposition applied to the term $(1+\tau)^{-1/2}$ in this formula results another form for expression (A.11)

$$\begin{aligned} F(A,B) &= K_0(2\sqrt{AB}) - \mathrm{Sign}(A-B)e^{-2\sqrt{AB}} \\ &\quad\times \sum_{n=0}^\infty c_n\frac{E_n(4\sqrt{AB}\tau_0)}{(4\sqrt{AB})^{n+1/2}}, \end{aligned} \tag{A.12}$$

where $c_0 = 1$ and

$$c_n = (-1)^n(1-\tfrac{1}{2})(1-\tfrac{1}{4})\cdots(1-\tfrac{1}{2n}) \quad (\text{for } n \geqslant 1). \tag{A.13}$$

The function $E_n(t) = \int_0^t e^{-q}q^{n-1/2}dq$ obeys the next recurrent formula

$$E_n(t) = (n-\tfrac{1}{2})E_{n-1}(t) - t^{n-1/2}e^{-t}. \tag{A.14}$$

One can see that

$$E_0(t) = \int_0^t e^{-q}q^{-1/2}dq = 2\int_0^{\sqrt{t}} e^{-p^2}dp = \sqrt{\pi}\,\mathrm{erf}(\sqrt{t}).$$

Another asymptotic expression was obtained for the case $\tau_0 \geqslant 1$. Here the term $(1+\tau)^{-1/2}$ in the formula (A.11) is represented as $(\tau(1+1/\tau))^{-1/2}$ and the series decomposition in inverse power of the value τ is used. It results in expression

$$F(A,B) = \begin{cases} S(A,B) & (A > B), \\ 2K_0(2\sqrt{AB}) - S(A,B) & (A < B), \end{cases} \tag{A.15}$$

where

$$S(A,B) = \mathrm{e}^{-2\sqrt{AB}} \sum_{n=0}^{\infty} \frac{c_n}{\tau_0^n} I_{n+1}(\tau_0 4\sqrt{AB}). \tag{A.16}$$

Note that here the same integral (A.8) is used and the coefficients c_n in the formulae (A.12) and (A.16) are the same.

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